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ALTERNATIVE FUZZY OPERATIONS: A CRITICAL APPROACH AND APPLICATIONS IN SOCIAL AND ECONOMIC SCIENCES

Abstract: *The applications of fuzzy numbers to Social Sciences, Economy, and Natural Sciences request, in various cases that the spreads, i.e. the region of indeterminateness, of the results of the operations between fuzzy numbers be less than the ones expected by the Zadeh's extension principle. Moreover, it appears to be necessary to consider operations that save the shapes of fuzzy numbers. To this aim fuzzy operations are dealt with, alternatives to the operations induced by the extension principle. Critical analyses of logical principles which support the various operations are carried out. Some application to Social Sciences and Economy are considered.*

Key words: *Modelling impreciseness, fuzzy numbers, fuzzy structures, applications to Social Sciences and Economy.*

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1. Introduction and motivation

Starting from the paper (Zadeh, 1965), Zadeh imposed the idea that a suitable tool to deal with the impreciseness, in the descriptions made in terms of natural language, are *fuzzy numbers*.

A major aim in developing modelling in fuzzy set context is just to obtain a "*satisfying*" *algebraic structure* for fuzzy numbers. The operations in the set of fuzzy numbers are usually obtained by the Zadeh extension principle (Zadeh, 1965, 1968, 1973, 1975a, 1975b, 1975c; Yager, 1986; Klir, Yuan, 1995).

The extension principle based operations present some drawbacks for the applications, e. g. to Social Sciences, Economy, Geology, both by an *algebraic* point of view and *logical* and *practical* aspects (Ban, Bede, 2003; Bede, Fodor, 2006; Grzegorzewski, Mrowska, 2005; Mares, 1997, 2001; Maturo, 2004, 2006, 2009a, 2009b), e.g.:

- the spreads of the sum is *fast increasing* with the repeated additions;
- the Zadeh's multiplication is *not distributive* with respect to the addition;
- the shape of fuzzy numbers is *not preserved* by multiplication.

Then, for the applications in the Natural, Social and Economic Sciences it is important to individuate some suitable variants of the classical fuzzy numbers

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operations that may interpret how the inexactness makes for greater efficiency, for producing significant meanings and comprehension. To this aim suitable alternative operations are here introduced.

2. The classical operations and order relations between compact intervals

Let ***C*** be the set of the compact intervals of R. For every intervals [a, b] and [c, d] in ***C***, we assume the following operations:

$$[a, b] + [c, d] = [a + c, b + d]; \tag{2.1}$$

$$[a, b] \cdot [c, d] = [\min\{ac, ad, bc, bd\}, \max\{ac, ad, bc, bd\}]. \tag{2.2}$$

The subtraction and division are also defined on ***C*** by the formulae:

$$[a, b] - [c, d] = [a, b] + [-b, -a]; \tag{2.3}$$

$$\text{if } 0 \notin [c, d], [a, b] / [c, d] = [a, b] \cdot [1/d, 1/c]. \tag{2.4}$$

The main algebraic properties of operations between compact intervals are the following (Klir, Yuan, 1995:103):

- The addition + defined by (2.1) is commutative, associative, having 0 = [0, 0] as neuter element.
- The multiplication defined by (2.2) is commutative, associative, having 1 = [1, 1] as neuter element.
- For every compact intervals [a, b], [c, d], [e, f] the following sub-distributive property holds:

$$([a, b] + [c, d])\,[e, f] \subseteq [a, b]\,[e, f] + [c, d]\,[e, f]. \tag{2.5}$$

- The distributivity, i.e. the equality

$$([a, b] + [c, d])\,[e, f] = [a, b]\,[e, f] + [c, d]\,[e, f], \tag{2.6}$$

holds iff $[a, b] \cdot [c, d] \geq 0$ or [e, f] is a degenerate interval.
The principal order relations on *C* are:

- The *main* order relation:

$$[a, b] \leq_M [c, d] \text{ if and only if } a \leq c \text{ and } b \leq d. \tag{2.7}$$

- The *strict* order relation**:**

$$[a, b] \leq_S [c, d] \text{ if and only if } b \leq c. \tag{2.8}$$

Both these relations are partial order relations, the second is contained in the first.

3. Fuzzy numbers: definitions and notations

The concept of fuzzy number borns as a generalization of the compact interval of R.

We consider compact intervals in R in which not all elements are equally important.

Definition 3.1 (Fuzzy number) Let [a, b] be a compact interval of R. A *fuzzy number* with *base* [a, b] is a function u: R $\to$ [0, 1], having as domain the set of real numbers and with values in [0, 1], such that:

(FN1) (*bounded support*) u(x) = 0 for x $\notin$ [a, b], and u(x) > 0 for x belonging to the open interval (a, b);

(FN2) (*compactness and normality*) for every r $\in$ (0, 1] the set $[u]^r = \{x \in R : u(x) \geq r\}$ is a nonempty *compact* interval.

The numbers a, b, are called respectively, the *left* and the *right endpoint* of u, and the set {x $\in$ [a, b] : u(x) > 0} is said to be the *support* of u, denoted S(u). Moreover, u is said to be *degenerate* if a = b, that is S(u) is a singleton.

Definition 3.2 (r-cuts) For every r such that $0 \leq r \leq 1$ the set $[u]^r = \{x \in [a, b] : u(x) \geq r\}$ is said to be the r-*cut* of u. The left and right endpoints of $[u]^r$ are denoted, respectively, u_λ^r and u_ρ^r. The fuzzy number u is said to be *simple* if c = d, that is C(u) is a singleton.

In particular, for r = 0, $[u]^0$ = [a, b], and, for r = 1, $[u]^1$ is a compact interval [c, d], called the *core* (or *central part*) of u, and denoted with C(u). The numbers c, d, are the *left* and the *right endpoint* of C(u).

Definition 3.3 (Spreads) The intervals [a, c) and (d, b] are, respectively, the *left part* and the *right part* of u. The real numbers L(u) = c – a, M(u) = d – c, and R(u) = b – d are, the *left*, *middle*, and *right spreads* of u. Their sum T(u) = b – a is the *total spread.*

Definition 3.4 (Relation $\subseteq$) Let u: R $\to$ [0, 1] and v: R $\to$ [0, 1] two fuzzy numbers. We say that u is contained in v, we write u $\subseteq$ v, if $u(x) \leq v(x)$, $\forall$ x $\in$ R.

Proposition 3.5 The relation $\subseteq$ is a partial order relation in the set of fuzzy numbers. A compact interval [a, b] is interpreted as a fuzzy set w: R $\to$ [0, 1] with base [a, b] and such that w(x) = 1, $\forall$ x $\in$ [a, b]. Then it is the maximum fuzzy number with base [a, b] with respect to the order relation $\subseteq$.

Notations 3.6 We assume the following notations:

- (*endpoints notation*) u ~ (a, c, d, b) stands u is a fuzzy number with endpoints a, b and core [c, d]; u ~ (a, c, b) for simple u;
- (*spreads notation*) u ~ [c, d, L, R] denotes that u is a fuzzy number with core [c, d] and left and right spreads L and R, respectively; u ~ [c, L, R] denotes simple u;
- (*r-cut spreads notation*) the numbers $L^r(u) = (c - u_\lambda^r)$ and $R^r(u) = (u_\rho^r - d)$ are called the *r-cut left spread* and the *r-cut right spread* of u, we write $[u]^r = [c, d, L^r(u), R^r(u)]$, and, if u is simple, we write also $[u]^r = [c, L^r(u), R^r(u)]$;
- (*sign*) the fuzzy number u ~ (a, c, d, b) is said to be *positive*, *strictly positive*, *negative*, or *strictly negative*, if $a \geq 0$, a > 0, $b \leq 0$, or b < 0, respectively;
- (*c-sign*) the fuzzy number u ~ (a, c, d, b) is said to be *c-positive*, *strictly c-positive*, *c-negative*, or *strictly c-negative*, if $c \geq 0$, c > 0, $d \leq 0$, or d < 0, respectively.

Definition 3.7 We say that the fuzzy number u ~ (a, c, d, b) is a *trapezoidal fuzzy number*, let us write u = (a, c, d, b), if:

$$\forall\, x \in [a, c),\quad a < c \;\Rightarrow\; u(x) = (x - a)/(c - a), \tag{3.1}$$

$$\forall\, x \in (d, b],\quad d < b \Rightarrow u(x) = (b - x) / (b - d). \tag{3.2}$$

A simple trapezoidal fuzzy number u = (a, c, c, b) is said to be a *triangular fuzzy number* and we write u = (a, c, b). A trapezoidal fuzzy number u = (c, c, d, d), with support equal to the core is said to be a *rectangular fuzzy number* and is identified with the compact interval [c, d] of R.

Proposition 3.8 In terms of r-cut left and right spreads u ~ [c, d, L, R] is a trapezoidal fuzzy number, we write u = [c, c', L, R], iff:

$$L^r(u) = (1 - r)\,(c - a) = (1 - r)\,L,\quad R^r(u) = (1 - r)\,(b - d) = (1 - r)\,R. \tag{3.3}$$

In particular, u ~ [c, L, R] is a triangular fuzzy number, we write u = [c, L, R], iff:

$$L^r(u) = (1 - r)\,(c - a) = (1 - r)\,L,\quad R^r(u) = (1 - r)\,(b - c) = (1 - r)\,R. \tag{3.4}$$

Several ordering can be defined in the set of fuzzy numbers (Klir, Yuan, 1995; Dubois, Prade, 1980, 1988; Mares, 1997, 2001; Maturo, 2009a, 2009b). We focus our attention on some fundamental orderings that play an important role when choices among social or economic actions are involved.

Let Φ be the set of all the fuzzy numbers.

Proposition 3.9 (Main order) The relation $\lesssim_M$ on Φ such that:

$$\forall\, u, v \in \Phi,\; u \lesssim_M v \Leftrightarrow \forall\, r \in [0, 1],\; [u]^r \le [v]^r, \tag{3.5}$$

is a *partial* ***order*** *relation* on Φ, called the *main order*.

The main order is the basic ordering in the set of fuzzy numbers whatever the shape.

Proposition 3.10 (Trapezoidal order) The relation $\lesssim_T$ on Φ such that:

$$\forall\, u, v \in \Phi,\; u \lesssim_T v \Leftrightarrow [u]^0 \le [v]^0,\; [u]^1 \le [v]^1 \tag{3.6}$$

is a *partial* ***preorder*** *relation* on Φ, called the *trapezoidal order*.

Such a relation is mainly useful if the trapezoidal shape is preferred, because of the relative simplicity in handling these numbers. The restriction of $\lesssim_T$ to the set **T** of the trapezoidal fuzzy numbers is a partial ***order*** relation.

Proposition 3.11 (Crisp order) The relation $\lesssim_C$ such that:

$$\forall\, u, v \in \Phi,\; u \lesssim_C v \Leftrightarrow [u]^1 \le [v]^1 \tag{3.7}$$

is a *partial* ***preorder*** *relation* on Φ, called the *core order* or *crisp order*.

The relation $\lesssim_C$ is useful when peripheral spreads are considered of marginal importance with respect to the central ones. The restriction of $\lesssim_C$ to the set ***C*** of the rectangular fuzzy numbers is the ***order*** relation (2.7). Moreover the restriction of $\lesssim_C$ to the set of simple fuzzy numbers is a ***total preorder*** *relation*.

Proposition 3.12 (Strict order) The relation $\lesssim_S$ such that:

$$\forall\, u, v \in \Phi,\ u \lesssim_C v \Leftrightarrow (x \in S(u),\, y \in S(v)) \Rightarrow x \le y \tag{3.8}$$

is a *partial* ***preorder*** *relation* on Φ, called the *strict order*. The restriction of $\lesssim_C$ to the set ***C*** of the rectangular fuzzy numbers is the ***order*** relation (2.8).

4. The extension principle operations

Definition 4.1 (Zadeh's extension principle operations) Let Ψ be the set of the fuzzy sets with domain R. If $*$ is an operation in R, then the extension of $*$ to Ψ with the *Zadeh's extension principle* is defined as follows. For every $u, v \in \Psi$, $z \in R$, let $(u * v)^{-1}(z) = \{(x, y) \in R \times R : x * y = z\}$. We define $u*v$ as the function

$$u * v : z \in R \to \begin{cases} 0 & \text{if } (u * v)^{-1}(z) = \varnothing \\ \sup\limits_{(x,y)\in(u*v)^{-1}(z)} \{\min\{u(x), v(y)\} & \text{if } (u * v)^{-1}(z) \neq \varnothing \end{cases}. \tag{4.1}$$

Remark 4.2 In general, if u and v are fuzzy numbers, we cannot conclude that $u * v$ is a fuzzy numbers. However, this occurs if $*$ is one of the usual arithmetic operations $+$, $-$, $\cdot$, $/$ (for the division, there must be the condition $0 \notin S(v)$).

If $u, v, u * v$ are fuzzy number, by (4.1) it follows:

$$\forall\, r \in [0, 1],\ [u * v]^r = [u]^r * [v]^r. \tag{4.2}$$

Then, (4.2) applied to the addition gives:

$$\forall\, r \in [0, 1],\ [u + v]^r = [u]^r + [v]^r = [u_\lambda^{\,r} + v_\lambda^{\,r},\ u_\rho^{\,r} + v_\rho^{\,r}]. \tag{4.3}$$

The (4.2) for the multiplication becomes, $\forall\, r \in [0, 1]$,

$$[u \cdot v]^r = [\min\{u_\lambda^{\,r} v_\lambda^{\,r}, u_\lambda^{\,r} v_\rho^{\,r}, u_\rho^{\,r} v_\lambda^{\,r}, u_\rho^{\,r} v_\rho^{\,r}\}, \max\{u_\lambda^{\,r} v_\lambda^{\,r}, u_\lambda^{\,r} v_\rho^{\,r}, u_\rho^{\,r} v_\lambda^{\,r}, u_\rho^{\,r} v_\rho^{\,r}\}]. \tag{4.4}$$

If u and v are positive then formula (4.4) reduces to:

$$\forall\, r \in [0, 1],\ [u \cdot v]^r = [u]^r \cdot [v]^r = [u_\lambda^{\,r} v_\lambda^{\,r},\ u_\rho^{\,r} v_\rho^{\,r}]. \tag{4.5}$$

By (4.2), (4.3), (4.4), and by the properties of addition and multiplications in the set ***C*** of compact intervals of R, it follows:

Proposition 4.3 The addition + defined by (4.3) is commutative, associative, having $0 = [0, 0]$ as neuter element; the multiplication defined by (4.4) is commutative, associative, having $1 = [1, 1]$ as neuter element. Moreover the multiplication is subdistributive with respect to the addition, i.e. for every fuzzy numbers u, v, w:

$$(u + v)\, w \subseteq u\, w + v\, w, \quad \text{(subdistributivity)} \tag{4.6}$$

where $\subseteq$ denotes inclusion between fuzzy sets; the distributivity holds iff u and v are both positive or both negative fuzzy numbers or w is a degenerate fuzzy number.

In terms of spread notation the addition is defined by formulae:

$$C(u + v) = C(u) + C(v); \quad (4.7)$$

$$\forall\, r \in [0, 1),\ L^r(u + v) = L^r(u) + L^r(v), \quad R^r(u + v) = R^r(u) + R^r(v). \quad (4.8)$$

If u and v are positive, in terms of spread notation the multiplication is defined by:

$$C(u \cdot v) = C(u) \cdot C(v); \quad (4.9)$$

$$\forall\, r \in [0, 1),\ L^r(u \cdot v) = u_\lambda^1\, v_\lambda^1 - u_\lambda^r\, v_\lambda^r = u_\lambda^1\, L^r(v) + v_\lambda^1\, L^r(u) - L^r(u)L^r(v); \quad (4.10)$$

$$\forall\, r \in [0, 1),\ R^r(u \cdot v) = u_\rho^r\, v_\rho^r - u_\rho^1\, v_\rho^1 = u_\rho^1\, R^r(v) + v_\rho^1\, R^r(u) + R^r(u)\, R^r(v). \quad (4.11)$$

Remark 4.4 (Case of trapezoidal numbers). For addition of trapezoidal (in particular triangular) fuzzy numbers, (4.8) is equivalent to the simpler formula:

$$L^0(u + v) = L^0(u) + L^0(v), \quad R^0(u + v) = R^0(u) + R^0(v). \quad (4.12)$$

For multiplication of trapezoidal fuzzy numbers, formulae (4.10) and (4.11) can be written:

$$\forall\, r \in [0, 1),\ L^r(u \cdot v) = (1 - r)\, u_\lambda^1\, L(v) + (1 - r)\, v_\lambda^1\, L(u) - (1 - r)^2\, L(u)\, L(v); \quad (4.13)$$

$$\forall\, r \in [0, 1),\ R^r(u \cdot v) = (1 - r)\, u_\rho^1\, R(v) + (1 - r)\, v_\rho^1\, R(u) + (1 - r)^2\, R(u)\, R(v). \quad (4.14)$$

Remark 4.5. Let $u \cdot_a v$ be the trapezoidal fuzzy number having the same core and the same support of the product $u \cdot v$. By (3.3), (4.10) and (4.11) for r = 0, it follows:

$$\forall\, r \in [0, 1),\ L^r(u \cdot_a v) = (1 - r)\, u_\lambda^1\, L(v) + (1 - r)\, v_\lambda^1\, L(u) - (1 - r)\, L(u)\, L(v); \quad (4.15)$$

$$\forall\, r \in [0, 1),\ R^r(u \cdot_a v) = (1 - r)\, u_\rho^1\, R(v) + (1 - r)\, v_\rho^1\, R(u) + (1 - r)\, R(u)\, R(v). \quad (4.16)$$

So, by comparing (4.13), (4.14) with (4.15), (4.16), respectively, it follows that, in general, $u \cdot v$ is not a trapezoidal fuzzy number since, for every $r \in (0, 1)$,

$$L^r(u \cdot v) \geq L^r(u \cdot_a v), \text{ the equality holds iff } L(u)L(v) = 0; \quad (4.17)$$

$$R^r(u \cdot v) \leq R^r(u \cdot_a v), \text{ the equality holds iff } R(u)R(v) = 0. \quad (4.18)$$

The Zadeh operations are compatible with the orders $\lesssim_M$, $\lesssim_T$, and $\lesssim_C$. Precisely, from the definitions in Sec. 3, and formulae (4.3), and (4.5) it follows:

Proposition 4.6 (Compatibility theorem). Let u, v, and w be fuzzy numbers. Then, for every $J \in \{M, T, C\}$, we have:

(CA) $u \lesssim_J v \Rightarrow u + w \lesssim_J v + w$ (*compatibility of* $\lesssim_J$ *w. r. to the addition*);

(CM) $0 \lesssim_J u, 0 \lesssim_J v \Rightarrow 0 \lesssim_J u \cdot v$ (*compatibility of* $\lesssim_J$ *w. r. to the multiplication*).

5. A critical analysis of the Zadeh's operations and alternative proposals

The definitions given by (4.1) or the equivalent ones obtained by (4.2) have some “drawbacks” for the practical applications in Social and Economic ambit.

(D1) The multiplication *is not distributive* with respect to the addition;

(D2) If we wish to consider special classes of fuzzy numbers as the trapezoidal or the triangular ones, they *are not closed with respect to the multiplication*, because the product of two trapezoidal fuzzy numbers in general is not trapezoidal.

(D3) The spread of the sum of two fuzzy numbers is the sum of the spreads, then by aggregating as "sum" fuzzy numbers we have *fast increasing spreads*. This is often in contradiction with the intuitive idea and the experience that many uncertainties can compensate each other.

(D4) The spreads of the product of two fuzzy numbers u and v are functions increasing of the extremes of the cores and the correspondent spreads of u and v. Then, if fuzzy numbers are aggregated by the "product", *too increasing indeterminateness* can result.

An important research work is thus to identify possible alternatives to the operations based on the extension principle. Below we analyse some variants of the classical addition and multiplication of fuzzy numbers. We call also "addition" and "multiplication", respectively, these variants, that have the most part of the properties of the corresponding classical operations, but not all the previous disadvantages.

(A1) *Approximate multiplication* (Grzegorzewski, Mrowska, 2005; Bede, Fodor 2006). If we wish consider only trapezoidal fuzzy numbers, then we can replace the Zadeh's product $u \cdot v$ with the trapezoidal fuzzy number $u \cdot_a v$ having the some support and the some core of $u \cdot v$.

(A2) *Cross multiplication* (Ban, Bede, 2003; Bede, Fodor 2006). Let u and v be two c-positive fuzzy numbers. The cross product $u \bullet v$ is the fuzzy number defined by formulae:

$$C(u \bullet v) = C(u) \cdot C(v); \tag{5.1}$$

$$\forall\ r \in [0, 1),\ L^r(u \bullet v) = u_\lambda^{\ 1}\, L^r(v) + v_\lambda^{\ 1}\, L^r(u); \tag{5.2}$$

$$\forall\ r \in [0, 1),\ R^r(u \bullet v) = u_\rho^{\ 1}\, R^r(v) + v_\rho^{\ 1}\, R^r(u). \tag{5.3}$$

Remark 5.1 The cross multiplication can be extended to the set Φ^* of all the fuzzy numbers that have not 0 as interior point of the core by utilizing the "signs rules"

$$(-u) \bullet v = u \bullet (-v) = -(u \bullet v).$$

In particular the cross multiplication can be extended to all the triangular fuzzy numbers.

Remark 5.2 The cross multiplication satisfies many important algebraic properties, in particular the *distributive property* $(u + v) \bullet w = u \bullet w + v \bullet w$ holds iff u and v are both c-positive or both c-negative or w is a degenerate fuzzy number, then in a stronger form with respect to the Zadeh multiplication.

(A3) *Bounded operations*. An addition $+_b$, and a multiplication $\cdot_b$, called b-addition and b-multiplication, respectively, are introduced, in such a way that the left, and right spreads of the sum $u +_b v$ and the product $u \cdot_b v$ are the maximum of the corresponding spreads of u and v.

Remark 5.3 The maximum property imply that the bounded operations are very appropriate operations when we have to deal with mathematical problems of convergence of sequences of fuzzy numbers.

In terms of spread notation the b-addition is defined by formulae:

$$C(u +_b v) = C(u) + C(v); \tag{5.4}$$

$$\forall\, r \in [0, 1),\ L^r(u +_b v) = \max\{L^r(u), L^r(v)\};\ \ R^r(u +_b v) = \max\{R^r(u), R^r(v)\}. \tag{5.5}$$

Moreover the b-multiplication is defined by formulae:

$$C(u \cdot_b v) = C(u) \cdot C(v); \tag{5.6}$$

$$\forall\, r \in [0, 1),\ L^r(u \cdot_b v) = \max\{L^r(u), L^r(v)\};\ \ R^r(u \cdot_b v) = \max\{R^r(u), R^r(v)\}. \tag{5.7}$$

Let us compare the Zadeh, approximate, and cross products of two c-positive trapezoidal fuzzy numbers u and v. They have the same core. The left r-spreads are:

$$\forall\, r \in [0, 1),\ L^r(u \cdot v) = u_\lambda^1\, L^r(v) + v_\lambda^1\, L^r(u) - L^r(u)\, L^r(v); \quad \text{(Zadeh product)} \tag{5.8}$$

$$\forall\, r \in [0, 1),\ L^r(u \cdot_a v) = u_\lambda^1\, L^r(v) + v_\lambda^1\, L^r(u) - L^r(u)\, L^r(v)/(1-r); \quad \text{(approx product)} \tag{5.9}$$

$$\forall\, r \in [0, 1),\ L^r(u \bullet v) = u_\lambda^1\, L^r(v) + v_\lambda^1\, L^r(u). \quad \text{(cross product)} \tag{5.10}$$

Then, for $r \in (0, 1)$ and $L^r(u)\, L^r(v) \neq 0$, the left r-spread of the approximate product is less than the one of the Zadeh product, while the left r-spread of the cross product is greater than the one of the Zadeh product.

The right r-spreads are:

$$\forall\, r \in [0, 1),\ R^r(u \cdot v) = u_\rho^1\, R^r(v) + v_\rho^1\, R^r(u) + R^r(u)\, R^r(v); \quad \text{(Zadeh product)} \tag{5.11}$$

$$\forall\, r \in [0, 1),\ R^r(u \cdot_a v) = u_\rho^1\, R^r(v) + v_\rho^1\, R^r(u) + R^r(u)\, R^r(v) / (1 - r); \text{ (approx product)} \tag{5.12}$$

$$\forall\, r \in [0, 1),\ R^r(u \bullet v) = u_\rho^1\, R^r(v) + v_\rho^1\, R^r(u). \quad \text{(cross product)} \tag{5.13}$$

Then, for $r \in (0, 1)$ and $R^r(u)R^r(v) \neq 0$, the right r-spread of the approximate product is greater than the one of the Zadeh product, while the left r-spread of the cross product is less than the one of the Zadeh product.

Remark 5.4 (Cross product properties) An advantage of the cross product is that the cross product of two trapezoidal fuzzy numbers is a trapezoidal fuzzy number; moreover the cross product of two triangular fuzzy numbers is a triangular fuzzy number. A disadvantage of the cross product is that the cross product of positive fuzzy numbers is not necessary a positive fuzzy number. As for compatibility with the order / preorder relations we have the following:

Proposition 5.5 Let u, v, and w be fuzzy numbers. Then, for every $J \in \{M, T, C\}$, we have:

(CMA) $0 \lesssim_J u,\ 0 \lesssim_J v \Rightarrow 0 \lesssim_J u \cdot_a v$ (*compatibility of* $\lesssim_J$ *w. r. to the approximate multiplication*);

(CMC) $0 \lesssim_C u,\ 0 \lesssim_C v \Rightarrow 0 \lesssim_C u \bullet v$ (*compatibility of* $\lesssim_C$ *w. r. to the cross multiplication*).

Remark 5.6 (Bounded operations properties) The main algebraic properties of the bounded operations are:

(B1) *Shape preserving property*. The b-sum and the b-product of two trapezoidal fuzzy numbers are trapezoidal fuzzy numbers. Moreover b-sum and the b- product of simple fuzzy numbers are simple fuzzy numbers.

(B2) *Semigroup property*. The b-addition and b-multiplication are associative, commutative, and having neuter elements 0 and 1, respectively.

(B3) *Distributivity property*. The b-addition is subdistributive with respect to the b-addition. That is, for every fuzzy numbers u, v, w, we have $(u +_b v) \cdot_b w \subseteq u \cdot_b w +_b v \cdot_b w$. The distributivity holds iff $C(u)\, C(v) \geq 0$ or $C(w)$ is a real number.

(B4) *Distributivity in the set of triangular fuzzy number*. The set Δ of triangular fuzzy number is closed with respect to the b-addition and the b-multiplication. Moreover, in Δ, the b-addition is distributive with respect to the b-addition.

(B5) *Spread limited*. Let $\{u_i\}_{i\in I}$ is a family of fuzzy numbers and m is a positive real number such that $\max\{L(u_i), R(u_i)\} \leq m$, $\forall\, i \in I$. If the fuzzy number u is a finite or countable b-sum or b-product of elements of $\{u_i\}_{i\in I}$, then $\max\{L(u), R(u)\} \leq m$.

(B6) *Compatibility with orders*. The b-addition is compatible with the orders $\lesssim_M$, $\lesssim_T$, and $\lesssim_C$, and the b-multiplication is compatible with $\lesssim_C$. Let u, v, and w be fuzzy numbers. For every $J \in \{M, T, C\}$, we have:

(BCA) $u \lesssim_J v \Rightarrow u +_b w \lesssim_J v +_b w$ (*compatibility of* $\lesssim_J$ *w. r. to the b-addition*);

(BCM) $0 \lesssim_C u,\ 0 \lesssim_C v \Rightarrow 0 \lesssim_C u \cdot_b v$ (*compatibility of* $\lesssim_C$ *w. r. to the b-multiplication*).

The properties of the b-addition and b-multiplication are useful for their utilization in modelling many social systems whenever we recognize that the indeterminateness should not be too increasing by aggregating fuzzy numbers.

Remark 5.7 A critical analysis of previous operations individuate the following drawbacks:

- The spreads of the b-product $u \cdot_b v$ don't consider the size and the sign of the fuzzy numbers u and v.
- Every spread of the cross multiplication $u \bullet v$ is a sum of two product of a size by a spread. Especially with repeated cross multiplications, we can have spreads very high.
- The b-product and the cross product of two positive fuzzy numbers is c-positive but not necessary positive.

Then we proposed in (Maturo, 2009a, 2009b), the following new alternative multiplication, that we called *cross maximum multiplication*. It can be a palatable compromise between the b-multiplication and the cross multiplication.

Definition 5.8 (Cross maximum multiplication). Let u and v be two c-positive fuzzy numbers. Let us define the "cross maximum" product $u \bullet_m v$ as the fuzzy number defined by formulae:

$$C(u \bullet_m v) = C(u) \cdot C(v); \tag{5.14}$$

$$\forall\ r \in [0, 1),\ L^r(u \bullet_m v) = \max\{u_\lambda^1\, L^r(v),\ v_\lambda^1\, L^r(u)\}; \quad (5.15)$$

$$\forall\ r \in [0, 1),\ R^r(u \bullet_m v) = \max\{u_\rho^1\, R^r(v),\ v_\rho^1\, R^r(u)\}. \quad (5.16)$$

The cross maximum multiplication permits to outweigh the disadvantages (D2) and (D4). As the cross multiplication, also the cross maximum multiplication can be extended to the set Φ* of all the fuzzy numbers that have not 0 as interior point of the core by utilizing the "signs rules".

The cross maximum multiplication is preserving the trapezoidal or triangular shape. In fact, (5.14), (5.15), and (5.16) imply the following:

Proposition 5.9 (Preserving shape theorem) The cross maximum product of two trapezoidal fuzzy numbers is a trapezoidal fuzzy number. Moreover the cross maximum product of two simple fuzzy numbers is a simple fuzzy number.

If u and v are positive trapezoidal fuzzy numbers, then, for every $r \in [0, 1)$, $u_\lambda^1 \geq L^r(u)$, and $v_\lambda^1 \geq L^r(v)$, and so, by (4.10), $L^r(u \bullet_m v) \leq L^r(u \cdot v)$.

Then, by proposition 4.4 it follows.

Proposition 5.10 (Compatibility with orders) Let u and v fuzzy numbers. Then, for every $J \in \{M, T, C\}$, we have the following property of *compatibility of* $\lesssim_J$ *w. r. to the cross maximum multiplication*:

(CMM) $0 \lesssim_J u,\ 0 \lesssim_J v \Rightarrow 0 \lesssim_J u \bullet_m v$.

6 Alternative operations based on t-norms and t-conorms

We propose two possible procedures for alternative operations based on t-norms and t-conorms. The first is to use a suitable t-norm instead of the operation of the minimum in the formula of the extension principle; the second is to calculate the spreads of a sum or a product of two fuzzy numbers using a t-conorm.

First procedure. Let ⊗ be a t-norm, i.e. an operation in [0, 1] associative, commutative, increasing with respect to every variable, and having 1 as neutral element. Particular cases are (Klir, Yuan, 1995:63):

(1) *Standard intersection* $\otimes_s$: $x \otimes_s y = \min\{x, y\}$;

(2) *Algebraic product* $\otimes_a$: $x \otimes_a y = x \cdot y$;

(3) *Bounded difference* $\otimes_b$: $x \otimes_b y = \max\{0, x + y - 1\}$;

(4) *Drastic intersection* $\otimes_d$: $x \otimes_d y = \min\{x, y\}$ if at least one between x, y is 1, and $x \otimes_d y = 0$ otherwise.

We have the following proposition:

Proposition 6.1 For every t-nom ⊗ and for every $x, y \in [0,1]$,

$$x \otimes_d y \leq x \otimes y \leq x \otimes_s y; \quad x \otimes_b y \leq x \otimes_a y. \quad (6.1)$$

Then, if in formula (4.1) of the extension principle operation, we replace min{u(x), v(u)} with $x \otimes y$, where ⊗ is a t-norm different from the standard intersection, we can reduce the spreads.

Second procedure. Let ⊕ be a t-conorm, i.e. an operation in [0, 1] associative, commutative, increasing with respect to every variable, and having 0 as neutral element (Sugeno, 1974; Weber, 1984; Klir, Yuan, 1995; Squillante, Ventre, 1998).

Particular cases are:

(1) *Standard union* $\oplus_s$: $x \oplus_s y = \max\{x, y\}$;

(2) *Algebraic sum* $\oplus_a$: $x \oplus_a y = x + y - x \cdot y$;

(3) *Bounded sum* $\oplus_b$: $x \oplus_b y = \min\{1, x + y\}$;

(4) *Drastic union* $\oplus_d$: $x \oplus_d y = \max\{x, y\}$ if at least one between x, y is 0, and $x \oplus_d y=1$ otherwise.

Proposition 6.2 For every t-conom $\oplus$ and for every x, y $\in$ [0,1],

$$x \oplus_s y \leq x \oplus y \leq x \oplus_d y; \quad x \oplus_a y \leq x \oplus_b y \tag{6.2}$$

We assume there exist two positive real numbers L_{max} and R_{max} that are the maximum left and right indeterminateness, respectively. Let S be the set of fuzzy numbers such that, for every $u \in S$, $L(u) \leq L_{max}$, and $R(u) \leq R_{max}$. Moreover let $\oplus$ be a t-conorm. Then we introduce an addition $+_\oplus$ and a multiplication $\cdot_\oplus$ on S, called $\oplus$-addition and $\oplus$-multiplication, respectively, in such a way the left and right indeterminateness of the sum $u +_\oplus v$ and the product $u \cdot_\oplus v$ are not greater than L_{max} and R_{max} respectively.

Definition 6.3 (Operations based on a t-conorm $\oplus$). We define the $\oplus$-addition on S by formulae:

$$C(u +_\oplus v) = C(u) + C(v); \tag{6.3}$$

$$\forall\, r \in [0, 1),\ L^r(u +_\oplus v) = [(L^r(u) / L_{max}) \oplus (L^r(v) / L_{max})]\, L_{max}; \tag{6.4}$$

$$\forall\, r \in [0, 1),\ R^r(u +_\oplus v) = [(R^r(u) / R_{max}) \oplus (R^r(v) / R_{max})]\, R_{max}. \tag{6.5}$$

Analogous formulae are introduced for multiplication:

$$C(u \cdot_\oplus v) = C(u) \cdot C(v); \tag{6.6}$$

$$\forall\, r \in [0, 1),\ L^r(u \cdot_\oplus v) = [(L^r(u) / L_{max}) \oplus (L^r(v) / L_{max})]\, L_{max}; \tag{6.7}$$

$$\forall\, r \in [0, 1),\ R^r(u \cdot_\oplus v) = [(R^r(u) / R_{max}) \oplus (R^r(v) / R_{max})]\, R_{max}. \tag{6.8}$$

By previous formulae it follows:

Proposition 6.4 (Semigroup properties). The $\oplus$-addition and $\oplus$-multiplication are associative, commutative, having neuter elements 0 and 1, respectively.

Remark 6.5. If $\oplus$ is the *standard union* then the $\oplus$-addition and the $\oplus$-multiplication reduce to the bounded operations. Moreover, if $\oplus$ is the *bounded sum*, i.e. $a \oplus b = \min\{1, a + b\}$, and the left and right spreads of the fuzzy numbers considered are very smaller then L_{max} and R_{max}, respectively, then the $\oplus$-addition and the $\oplus$-multiplication reduce to the Zadeh operations.

Proposition 6.6 (Compatibility property). Let $\oplus$ be a t-conorm such that, for every h, a, b $\in$ [0, 1] with $h + a \leq 1$, we have $(h + a) \oplus b \leq h + (a \oplus b)$. Then, for every $J \in \{M, T, C\}$, and for every fuzzy numbers u, v, and w belonging to S, we have:

(CCA) $u \lesssim_J v \Rightarrow u +_\oplus w \lesssim_J v +_\oplus w$ (*compatibility of* $\lesssim_J$ *w. r. to the* $\oplus$*-addition*);

Remark 6.7. The condition $\forall$ h, a, b $\in$ [0, 1]: (h + a $\leq$ 1) $\Rightarrow$ (h + a) $\oplus$ b $\leq$ h + (a $\oplus$ b) is satisfied if a $\oplus$ b = min{1, a + b} (*bounded sum*) or a $\oplus$ b = max{a, b} (*standard union*).

Then, if $\oplus$ is the *bounded sum or the standard union* then the compatibility property (CCA) holds.

7. Some applications to Social and Economic Sciences

The most recent trends of research on Social and Economic Sciences is based on the idea that individuals make social choices with personal interpretations of their social roles (Boudon, 1967; 1969). Then the capability of every individual to interpret a social role and the valuation of the global effect of the actions of a group of persons is a very fundamental problem. In particular there is the problem that "perverse effects" can happen as a sum of rational, but not coordinate, behaviours of the individuals belonging to the group. The mathematical tools to evaluate the interaction of decisions of many individuals are decision theory and game theory (Boudon, 1967; 1969; Lindley, 1971; Mark 1994; Maturo, Tofan, Ventre, 2004; Mares, 2001).

The uncertainty on the occurrence of the events that constitute the states of nature is controlled by the subjective probability (de Finetti, 1970; Dubins, 1975; Lindley, 1971) and the uncertainty semantics from the theory of fuzzy sets, in particular by fuzzy numbers (Zadeh, 1965; Goguen, 1974; Bellman, Giertz, 1973; Bouchon-Meunier, 1993; Maturo, Ventre, 2008).

A model of social appropriateness.

We consider the problem to evaluate the appropriateness of the individuals belonging to a set U *to make a certain social action* or *to have a particular social role*.

We assume that such appropriateness is defined by:

- a set $\mathbf{X} = \{X_1, X_2, \ldots, X_n\}$ of characteristics;
- a set of weights $\lambda = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$, where λ_j measures the importance of X_j for the social action or the social role considered,

with the following normalization conditions:

(C1) every λ_j is a trapezoidal fuzzy number (in particular a triangular one) with support $S(\lambda_j)$ contained in the interval [0, 1];

(C2) the sum of the cores of the fuzzy numbers λ_j contains 1, i.e.

$$C(\lambda_1) + C(\lambda_2) + \ldots + C(\lambda_n) \supseteq 1. \tag{7.1}$$

In particular, if the λ_j are triangular fuzzy numbers then the conditions (C1) and (C2) reduces to:

(C1T) every λ_j is a triangular fuzzy number with support $S(\lambda_j) \subseteq [0, 1]$;

(C2T) the sum of the cores of the fuzzy numbers λ_j is equal to 1, that is

$$C(\lambda_1) + C(\lambda_2) + \ldots + C(\lambda_n) = 1. \tag{7.2}$$

We can assume two different points of view:

(1) ***the objective evaluation***: an *expert* or a *committee of experts* associates to every pair ($x \in U$, $X_j \in \mathbf{X}$) a trapezoidal fuzzy number $\alpha_j(x) \subseteq [0, 1]$ that measures the degree in which x *satisfies* the characteristic X_j;

(2) ***the subjective evaluation***: each individual x associates to every characteristic X_j a trapezoidal fuzzy number $\alpha_j(x) \subseteq [0, 1]$ that measures the degree in which x *believes* himself satisfies the characteristic X_j.

The individual evaluation. The *objective evaluation* is necessary if a committee has the task to select individuals to do a job, the *subjective evaluation* can be important for a decision of a purchase, assuming a model in which the X_j are the functional characters of an object that x can buy and $\alpha_j(x)$ is the degree in which the individual x believes to need of X_j.

In order to evaluate the objective suitability or the subjective sense of appropriateness of the elements of U, we have to obtain, for every individual x, a global score $\gamma(x)$. At this aim a "multiplication" $\otimes$ and an "addition" $\oplus$ on the interval [0, 1] must be introduced such that:

- the product $\beta_j(x) = \lambda_j \otimes \alpha_j(x)$ is the *weighed measure* of the degree in which x satisfies to the characteristic X_j;
- the sum $\gamma(x) = \beta_1(x) \oplus \beta_2(x) \oplus \ldots \oplus \beta_n(x)$ is the *global score* of the individual x.

The multiplication $\otimes$ cannot be the Zadeh's extension principle one, because, in general, the Zadeh product of trapezoidal fuzzy numbers is not a trapezoidal fuzzy number. We have to assume a variant of the Zadeh multiplication that preserves the trapezoidal shape and such that every product $\beta_j(x) = \lambda_j \otimes \alpha_j(x)$ is contained in [0, 1]. For instance, we can assume $\otimes$ be the approximate, or the cross, or the cross maximum multiplication, considered in previous Sec. 5.

The addition can be the usual Zadeh addition, but, if we have to consider many characteristics X_j, it is preferable to have a global score with moderate spreads and then we have to consider an alternative addition, e. g. the b-addition or an addition based on a t-conorm $\oplus$.

Team evaluation. We can wish to give a global evaluation of a subset S of U, e. g. a subset that individuates a team for a job, in which every individual has a definite responsibility.

Let $S = \{x_1, x_2, \ldots, x_m\}$ and let μ_i be a positive trapezoidal fuzzy number that defines the amount of the tasks of x_i. The team S can be evaluate with the formula

$$v(S) = \mu_1 \otimes \gamma(x_1) \oplus \mu_2 \otimes \gamma(x_2) + \ldots + \mu_m \otimes \gamma(x_m), \tag{7.3}$$

where $\otimes$ and $\oplus$ are suitable alternative multiplication and addition, respectively.

8. Operations defined up to equivalence relations

Let S be a set of fuzzy numbers, and let $\sim$ be an equivalence relation $\sim$ on S. For every $u \in S$, let [u] denote the equivalence class of u.

Let + be an operation defined on S. The relation $\sim$ is compatible with the operation + if, for every u, u', v, v'$\in$S, the following implication holds:

$$([u] = [u'], [v] = [v']) \Rightarrow [u + v] = [u' + v']. \quad (8.1)$$

Let $*$ be a scalar multiplication on S, that is a function $*$: R $\times$ S $\rightarrow$ S. The relation $\sim$ is compatible with $*$ if, for every $\lambda \in$ R, u, u' $\in$ S,

$$([u] = [u']) \Rightarrow ([\lambda * u] = [\lambda * u']). \quad (8.2)$$

If $\sim$ is compatible with + and $*$, we can consider the induced operations $\mathbf{+}$ and $\mathbf{*}$ on S/$\sim$, defined as:

$$\forall\ [u], [v] \in S / \sim, [u] \mathbf{+} [v] = [u + v];$$
$$\forall\ \lambda \in R, [u] \in S / \sim, \lambda \mathbf{*} [u] = [\lambda * u].$$

Spread compensation. A particular important case is the equivalence $\sim_{sc}$ in the set Φ of all the fuzzy numbers, defined as:

$$u \sim_{sc} v \text{ iff } (C(u) = C(v) \text{ and } \forall\ r \in [0, 1), L^r(u) - R^r(u) = L^r(v) - R^r(v)). \quad (8.3)$$

We call this relation *spread compensation*. Some properties are the following.

Proposition 8.1 (Compatibility with the Zadeh's operations). The spread compensation $\sim_{sc}$ is compatible with the Zadeh's addition and the scalar multiplication of a fuzzy number by a real number.

Proposition 8.2 (Case of trapezoidal fuzzy numbers). The relation $\sim$ is the restriction of $\sim_{sc}$ to the set **T** of trapezoidal fuzzy numbers if and only if, for every u, v $\in$ T,

$$u \sim v \ \text{ iff } \ (C(u) = C(v) \text{ and } L^0(u) - R^0(u) = L^0(v) - R^0(v)). \quad (8.4)$$

Proposition 8.3 (Minimum property). If u$\in\Phi$ is such that f(r) = $L^r(u) - R^r(u)$ is a function decreasing of r, then the fuzzy number u* defined as:

$$C(u^*) = C(u), L^r(u^*) = L^r(u) - R^r(u), R^r(u^*) = 0$$

is the minimum of [u] with respect the inclusion $\subseteq$, i.e. u* $\subseteq$ v, for every v $\in$ [u].

If u $\in \Phi$ is such that f(r) = $L^r(u) - R^r(u)$ is a function increasing of r, then the fuzzy number u* defined as:

$$C(u^*) = C(u), L^r(u^*) = 0, R^r(u^*) = R^r(u) - L^r(u)$$

is the minimum of [u] with respect the inclusion $\subseteq$. .

From previous propositions, it follows:

Proposition 8.4 (Case of trapezoidal fuzzy numbers). If u is a trapezoidal fuzzy number, then there exists a trapezoidal fuzzy number u* $\in$ [u] such that u* $\subseteq$ v, for every v $\in$ [u].

Proposition 8.5 (Main algebraic properties). If Δ is the set of triangular fuzzy numbers then the quotient set $\Delta/\sim_{sc}$, with the addition $\mathbf{+}$ induced by the Zadeh's addition, is a *group*. If $\mathbf{*}$ is the operation induced by the scalar multiplication of a trapezoidal fuzzy number by a real number, then $(\Delta/\sim_{sc}, +, *)$ is a *vector space*.

Critical remarks 8.8 For every equivalence relation ~, all the fuzzy numbers belonging to an equivalence class represent the same concept by a suitable point of view. It can happen that, for every equivalence class [u], we have a particular meaningful fuzzy number u* belonging to the class, called *the normal form* of [u], obtained, e.g. by a minimization condition. For instance, if ~ is the spread compensation, and S is the set of trapezoidal fuzzy numbers, then the normal form of [u] is the minimal element of [u].

The consideration of the previous *up to equivalence operations* in the set of triangular fuzzy numbers, together with the *approximate multiplication*, attains the aims to avoid the disadvantages (D2) and (D3) and to get good algebraic structures in the set of triangular of fuzzy numbers.

The necessity to obtain a vector space in order to extend to triangular fuzzy random numbers the probabilistic concept of coherence is emphasised in (Maturo, 2004; Maturo, Ventre, 2008). A fuzzy extension of the subjective probability is considered in (Maturo, 2000, 2006).

9. Hyperstructures of fuzzy numbers and conclusions

A totally different and unifying point of view is to replacing the Zadeh or the alternative operations with particular hyperoperations. For the concept of hyperoperations, see, e.g., (Corsini, 1993; Maturo, 2001; Corsini, Leoreanu, 2003).

By utilizing the concept of hyperoperation we can unify the Zadeh's addition and all the introduced alternative additions. Each of these additions can be considered as a restriction of a particular hyperaddition •. Moreover Φ, T, and some other subsets of Φ, are commutative hypergroups with respect to •.

In particular this permits to have many interesting applications for the fuzzy extensions of the concept of probability (Maturo, 2000; 2001).

Two hyperoperations in the set Δ of triangular fuzzy numbers, and associated to the set of possible addition, are the following:

(H1) **hyperoperation σ**. If u and v are two triangular fuzzy numbers, then u σ v is the set of all the triangular fuzzy numbers w such that C(w) = C(u) + C(v) and:

$$L(w) \in [\max\{L(u), L(v)\}, L(u) + L(v)];\ R(w) \in [\max\{R(u), R(v)\}, R(u) + R(v)]. \quad (9.1)$$

(H2) **hyperoperation δ**. If u and v are two triangular fuzzy numbers, then u δ v is the set of all the triangular fuzzy numbers w such that C(w) = C(u) + C(v) and:

$$L(w) \in [\min\{L(u), L(v)\}, L(u) + L(v)];\ R(w) \in [\min\{R(u), R(v)\}, R(u) + R(v)]. \quad (9.2)$$

Remark 9.1. The set u σ v contains all the results of the Zadeh or alternative operation with spreads non superior to the Zadeh ones. But, for every u, v ∈ Δ, there exists $x \in \Delta$ such that $v \in u \, \sigma \, x$ if and only if $L(u) \leq L(v)$ and $R(u) \leq R(v)$. Then (Δ, σ) is not a quasihypergroup.

Remark 9.2. For every u, $v \in \Delta$, there exists $x \in \Delta$ such that $v \in u \, \delta \, x$. It is sufficient to assume $x = [C(x), L(x), R(x)]$, with $C(x) = C(v) - C(u)$, $L(x) \in [\min\{L(u), L(v)\}, L(v)]$, $R(x) \in [\min\{R(u), R(v)\}, R(v)]$.

Then we have the following theorem.

Theorem 9.3 (Δ, σ) and (Δ, δ) are two commutative semihypergroups, with (Δ, δ) an extension of (Δ, σ). Moreover

- (Δ, δ) is a hypergroup;
- (Δ, σ) is not a hypergroup.

Conclusions. We believe that the point of view based on hyperoperations can be the starting point for fruitful research. Indeed, it takes account of a dual type of uncertainty, the one on the numeric data, considering fuzzy numbers, and the other on the uncertainty of the result of an operation between fuzzy numbers, taken into account by hyperoperations.

References

Ban A. I., Bede B., (2003), *Cross product of L-R fuzzy numbers and properties*, Annals of Oradea University, Fasc. Matem. 9, 95-108

Bede B., Fodor J., (2006), *Product type operations between fuzzy numbers and their applications in Geology*, Acta Polytechnica Hungarica, Vol. 3, No 1, 123-139

Bellman R., Giertz M., (1973), *On the analytic formalism of the theory of fuzzy sets*, Information Sciences 5, 149-157

Boudon R., (1967), *L'analyse mathématique des fait socials*, Paris, Plon.

Boudon R., (1969), *Les methods in Sociologie,* Paris, P.U.F.

Bouchon-Meunier B., 1993, *La logique flou*, Presses Universitaires de France, Paris.

Corsini P., (1993), Prolegomena of hypergroup theory, Aviani Editore, Tricesimo.

Corsini P. and Leoreanu V., (2003), Applications of hyperstructure theory, Kluwer Academic Publishers.

de Finetti B., (1974), *Theory of Probability*, J. Wiley, New York.

Dubois, D., Prade H., (1980), *Fuzzy Set and Systems*, Academic Press, New York.

Dubois, D., Prade H., (1988), *Fuzzy numbers: An overview*. In Analysis of Fuzzy Information (J. C. Bedzek, Ed.), CRC-Press, Boca Raton, Vol 2, 3-39.

Dubins L. E., (1975), *Finitely additive conditional probabilities, conglomerability and disintegrations*, The Annals of Probability, 3, 89-99.

Goguen J., (1974), *On fuzzy robot planning*, Memo no. 1 on Artificial Intelligence, University of California, Los Angeles.

Grzegorzewski P., Mrowska E., (2005), *Trapezoidal approximations of fuzzy numbers*, Fuzzy Sets and Systems, 153 (2005), 115 – 135.

Klir G, Yuan B., (1995), *Fuzzy sets and fuzzy logic: Theory and Applications*, Prentice Hall, New Jersey.

Lindley D. V., (1971), *Making decisions*, J. Wiley, New York.

Mark J. G., (1994), *A primer on Decision Making. How decision Happen*, New York, The Free Press.

Mares M., (1997), *Weak arithmetic on fuzzy numbers*. Fuzzy sets and Systems, 91, 2, 143-154

Mares M., (2001), *Fuzzy Cooperative Games*, Springer Verlag, New York.

Maturo A., (2000), *Fuzzy events and their probability assessments*, Journal of Discrete Mathematical Sciences & Cryptography, Vol. 3, Nos 1-3, 83-94.

Maturo A., (2001), *Finitely additive conditional probabilities with values on a hyperstructure*, in "Journal of Information and Optimization Sciences", Analitic Publ. Delhi, Vol. 22, No 2, 227-240.

Maturo A., (2004), *Grandezze aleatorie fuzzy e loro previsioni per le decisioni in condizione di informazione parziale*, Current Topics in Computer Sciences, Cortellini and Luchian Ed., Panfilus, Iasi, 15-24.

Maturo A., (2006), *Fuzzy Conditional probabilities by the Subjective Point of View*, in Advances in Mathematics of Uncertainty, Tofan Ed., Performantica, Iasi, pp. 99-108.

Maturo A., (2009a), *Alternative fuzzy operations and applications to social sciences*, International Journal of Intelligent Systems, Vol. 24,1243-1264.

Maturo A., (2009b), *On some structures of fuzzy numbers*, Iranian Journal of Fuzzy Systems, Vol. 6, No 4, pp.49-59.

Maturo A., Tofan I., Ventre A., (2004), *Fuzzy Games and Coherent Fuzzy Previsions*, in Fuzzy Systems & A. I., 10, No 3, pp. 109-116.

Maturo A. and Ventre A., (2008), *On some extensions of the de Finetti coherent prevision in a fuzzy ambit*, Journal of Basic Science, 4(1), 95-103.

Squillante M., Ventre A. G. S., (1998), *Consistency for uncertainty measure*, International Journal of Intelligent Systems, vol 13, 419-430.

Sugeno M., (1974), *Theory of fuzzy integral and its Applications*, Ph. D. Thesis, Tokyo.

Weber S., (1984), *Decomposable measures and integrals for archimedean t-conorms,* J. Math. Anal. Appl. 101 (1), 114-138.

Yager R., (1986), *A characterization of the extension principle*, Fuzzy Sets Systems, 18, 3, 205-217.

Zadeh L., (1965), *Fuzzy sets*, Inf. Control, **8**, 338-353.

Zadeh L., (1968), *Probability measures of fuzzy events*, J. Math. Anal. Appl. 23, 421-427.

Zadeh L., (1973), *Outline of a new approach to the analysis of complex systems and decision processes*, IEEE Transactions on Systems, Man and Cybernetics, SMC 3, 28-44.

Zadeh L., (1975a), *The Concept of a Linguistic Variable and its Application to Approximate Reasoning I*, Information Sciences, 8, 199-249.

Zadeh L., (1975b), *The concept of a Linguistic Variable and its Applications to Approximate reasoning II*, Information Sciences 8, 301-357.

Zadeh L., (1975c), *The concept of a Linguistic Variable and its Applications to Approximate reasoning III*, Information Sciences 9, 43-80.