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THE DEVELOPMENT OF A SOFTWARE DEDICATED TO SOLVING THE FUZZY LINIAR PROGRAMS

Abstract: *A fuzzy linear program is a linear program in which the input parameters are mathematically modeled with fuzzy numbers. The benefit vector, the unknown vector, the constraint matrix, and the free terms vector composing a fuzzy linear program, all have fuzzy number components.*

The article develops algorithmic steps with which one can make a software for solving fuzzy linear programs. Even if the triangular fuzzy numbers were used in the example presented, the algorithm is valid for all polygonal fuzzy numbers.

In practice elementary fuzzy numbers such as rectangular, triangular and trapezoidal fuzzy numbers can be used; the medium fuzzy numbers such as the hexagonal and octagonal fuzzy numbers or the large fuzzy numbers such as the decagonal, dodecagonal, fuzzy numbers can also be used.

By using fuzzy numbers in linear programming, answers are provided to practical problems in a more realistic manner.

Key words: *linear programming, simplex algorithm, triangular fuzzy numbers.*

JEL: *D81, M15*

Elementary arithmetic operations on the set of triangular fuzzy numbers

Since the mid-twentieth century, the specialized literature in the field of fuzzy theory and applications in economics has gained a special foothold.

We will mention here a limited number of papers with a special impact in the field, works from which one can consult many other notions of the fuzzy theory: [Kaufmann, 1973], [Negoiţă, 1974] [Moisil, 1975], [Vlădeanu, 2004], [Bojadziev, 2006] and [Gherasim, 2014].

A ***triangular fuzzy number*** is an ordered triplet of real numbers:

$$\widetilde{a} = (a_1, a_2, a_3),\ a_1 \le a_2 \le a_3 \qquad (1)$$

The indicators associated with ***a triangular fuzzy number*** $\widetilde{a} = (a_1, a_2, a_3)$ are defined as follows:

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the core and middle of support	$a_N = a_2$; $\quad a_{Sp} = \dfrac{a_1 + a_3}{2}$	
the length of the support	$L_a^{Sp} = a_3 - a_1$	
the center of gravity	$a_G = < \widetilde{a} > = \dfrac{a_1 + 2 \cdot a_2 + a_3}{4}$	(2)
the sign	$\delta_a = \begin{cases} \text{sign}(a_G) & , a_G \neq 0 \\ \text{sign}(a_N) & , a_G = 0 \end{cases}$	

Defining some arithmetic operations on the set of the triangular fuzzy numbers that retain associated indicators leads to major advantages.

The* basic *arithmetic operations* with *triangular fuzzy numbers [Gherasim, 2014] are performed based upon the following relations:

$\widetilde{a} = (a_1, a_2, a_3)$, $\widetilde{b} = (b_1, b_2, b_3)$ $\quad , \widetilde{a}, \widetilde{b} \in \mathbf{F}_{tr}$		
Scalar multiplication	$t\widetilde{a} = \begin{cases} (ta_1, ta_2, ta_3) & , t \geq 0 \\ (ta_3, ta_2, ta_1) & , t < 0 \end{cases}$	
Addition and Subtraction	$\widetilde{a} + \widetilde{b} = (a_1 + b_1, a_2 + b_2, a_3 + b_3)$; $\widetilde{a} - \widetilde{b} = (a_1 - b_3, a_2 - b_2, a_3 - b_1)$	(3)
Multiplication and Division	$\widetilde{a}\widetilde{b} = \dfrac{a_G \cdot \widetilde{b} + \widetilde{a} \cdot b_G}{2}$; $\quad \dfrac{\widetilde{a}}{\widetilde{b}} = \dfrac{a_G \cdot \widetilde{b} + \widetilde{a} \cdot b_G}{2 \cdot b_G^2}, (\forall) b_G \neq 0$	

The ordering of triangular fuzzy numbers is obtained by successively and exclusively applying four ordering criteria:

O1:	$a_G < b_G \Rightarrow \widetilde{a} \prec \widetilde{b}$	
O2:	$a_N < b_N \Rightarrow \widetilde{a} \prec \widetilde{b}$	(4)
O3:	$\delta_a \cdot L_a^{Sp} < \delta_b \cdot L_b^{Sp} \Rightarrow \widetilde{a} \prec \widetilde{b}$	
O4:	$L_a^{Sp} < L_b^{Sp} \Rightarrow \widetilde{a} \prec\prec \widetilde{b}$	

Example of operations with triangular fuzzy numbers:

Let us consider the triangular fuzzy numbers: $\widetilde{a} = (5, 8, 11)$, $\widetilde{b} = (4, 9, 10)$ şi $\widetilde{c} = (0, 9, 10)$.

$$< \widetilde{a} > = \frac{5 + 2 \cdot 8 + 11}{4} = \frac{32}{4} = 8 \qquad < \widetilde{b} > = \frac{4 + 2 \cdot 9 + 10}{4} = \frac{32}{4} = 8 = < \widetilde{a} >$$

The weight center criterion (O1) does not decide on the order of the fuzzy numbers $\widetilde{a}$ and $\widetilde{b}$.

The criterion O2 (comparison of peaks) is applied:

$$a_N = 8 < 9 = b_N \quad \Rightarrow \quad \tilde{a} \prec \tilde{b} \qquad (5,8,11) \prec (4,9,10)$$

$$< \tilde{c} > = \frac{0 + 2\cdot 9 + 10}{4} = \frac{28}{4} = 7 < 8 = < \tilde{a} > = < \tilde{b} > \quad \Rightarrow \quad \tilde{c} \prec \tilde{a} \ \wedge \ \tilde{c} \prec \tilde{b}$$

$$\tilde{c} \prec \tilde{a} \prec \tilde{b} \qquad (0,9,10) \prec (5,8,11) \prec (4,9,10)$$

$$\tilde{a} + \tilde{b} = (5+4, 8+9, 11+10) = (9,17,21) \quad < \tilde{a} + \tilde{b} > = \frac{9 + 2\cdot 17 + 21}{4} = 16 = 8 + 8 = < \tilde{a} > + < \tilde{b} >$$

$$\tilde{a} - \tilde{b} = (5-10, 8-9, 11-4) = (-5,-1,7), \quad < \tilde{a} - \tilde{b} > = \frac{-5 + 2\cdot(-1) + 7}{4} = 0 = < \tilde{a} > - < \tilde{b} >$$

$$\tilde{a} \cdot \tilde{b} = \frac{8\cdot(5,8,11) + 8\cdot(4,9,10)}{2} = \frac{(40,64,88) + (32,72,80)}{2} = \frac{(72,136,168)}{2} = (36,68,84)$$

$$< \tilde{a} \cdot \tilde{b} > = \frac{36 + 2\cdot 68 + 84}{2} = 64 = 8 \cdot 8 = < \tilde{a} > \cdot < \tilde{b} >$$

$$\frac{\tilde{a}}{\tilde{b}} = \frac{\tilde{a}\cdot\tilde{b}}{< \tilde{b} >^2} = \frac{(36,68,84)}{8^2} = \left(\frac{9}{16}, \frac{17}{16}, \frac{21}{16}\right), \quad \left\langle \frac{\tilde{a}}{\tilde{b}} \right\rangle = \frac{\frac{9}{16} + 2\cdot\frac{17}{16} + \frac{21}{16}}{4} = \frac{64}{64} = 1 = \frac{8}{8} = \frac{< \tilde{a} >}{< \tilde{b} >}$$

Solving the linear programs (PL) by the Simplex algorithm with the penalty method

The linear programming has emerged as a distinct scientific discipline in the middle of the last century.

The first papers with a uniform and complete treatment were published by L. Kantorovici (1939) and F. Hitchcock (1941). In 1947 G. Dantzig and J. Von Newmann created the simplex method for solving linear programming problems.

Amongst the first reference works in the field of linear programming at international level we mention: [Baumol, 1963], [Dantzig, 1963] and [Gass, 1958].

In Romania, entire generations of professors and researchers approached the field of linear programming. Chronologically, we mention the volumes of: Boroş, 1970], [Mihoc, 1973], [Maliţa, 1975], [Drăgan, 1976], [Boldur, 1979], [Cerchez, 1982], [Purcaru, 1982] etc.

A ***linear program (PL)*** is composed of three groups of mathematical relations:

$$(PL) \left| \begin{array}{ll} (a) & [\min/\max]\ z_{(C,X)} = c_1x_1 + c_2x_2 + ... + c_nx_n \\ \hline (b) & \begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \le d_1 \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n = d_2 \\ a_{31}x_1 + a_{32}x_2 + ... + a_{3n}x_n \ge d_3 \ \ ... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \overset{\le}{\underset{\ge}{=}} d_m \end{cases} \\ (c) & x_j \ge 0 \ \ ,(\forall)j = \overline{1,n} \end{array} \right. \qquad (5)$$

The relation (a) contains ***the objective function*** of optimization $\mathbf{z}_{(C,X)}$ composed with ***the benefit vector*** $\mathbf{C} = (c_1, c_2, ..., c_n)$ and ***the unknown vector*** $\mathbf{X}^T = (x_1, x_2, ..., x_n)$.

The second group of relations (b) represents the restrictions imposed on the unknown to be determined and contains the ***matrix of restrictions*** $\mathbf{A} = (a_{ij})_{i=\overline{1,m}, j=\overline{1,n}}$ and the ***free terms vector*** $\mathbf{D}^T = (d_1, d_2, ..., d_m)$. The relations (c) are ***the conditions of non-negativity*** imposed on the unknown.

Solving a linear program (PL) consists of determining the positive values $x_j \geq 0$ (c) that comply with the restrictions (b) and optimizing the value $z_{(C,X)}$ of the objective function (a).

Among the many methods for solving linear programming problems (PL), the "***Simplex algorithm***" method was aggressively imposed, a method presented in detail in most of the aforementioned specialized volumes.

The general method of solving a linear program, called ***the simplex algorithm*** with ***the penalty method***, will be presented briefly with algorithmic steps.

Step 1. Obtaining ***a*** minimum ***initial admissible primary program***

- The first action from this step involves ***converting the optimum from maximum to minimum*** (if applicable). Thus, if we note w = –z and calculate w^* = min w then z^* = max z = $-w^*$. Therefore, in the linear program we will have either an initial $[\min]z$ or $[\min]w$.
- The second action involves ***changing the sign of the negative free terms*** $d_i < 0$ (if any), by multiplying both members of the respective restrictions by –1 (an action which reverses the meaning of inequality $\leq \longleftrightarrow \geq$).
- The third action involves ***the transformation of the inequality restrictions into the equality restrictions*** by adding to the inequalities $\leq$ the decrease to the inequalities respectively $\geq$ of a positive variable aslo called ***deviation variables***.
- The fourth action, also called ***the penalty method***, generates an initially ***allowable solution***, by adding a ***penalty variable*** to each restriction (the deviation variables with the positive sign also play the role of penalty variables)

$$(\text{PPA}) \quad \left| \begin{array}{l} \min z = c_1x_1 + c_2x_2 + ... + c_nx_n + M \cdot (x_{n+k+1} + x_{n+k+2} + ... + x_{n+k+m}) \\ \hline \begin{cases} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n + x_{n+1} \qquad + x_{n+k+1} \qquad = d_1 \\ a_{21}x_1 + \tilde{a}_{22}\tilde{x}_2 + ... + a_{2n}x_n \qquad - x_{n+2} \qquad + x_{n+k+2} \qquad = d_2 \\ \qquad ... \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \qquad + x_{n+k+m} = d_m \end{cases} \\ x_j \geq 0 \ , \forall j = \overline{1, n+k+m} ; \quad b_i \geq 0 \ , \forall i = \overline{1,m} \end{array} \right. \qquad (6)$$

In the objective function, the most m variables added have ***a high enough penalty coefficient*** $M>0$. For example, M can be chosen as follows:

$$A_{\max}=\max_{\substack{i=\overline{1,m}\\ j=\overline{1,n}}}|a_{ij}|,\quad C_{\max}=\max_{j=\overline{1,n}}|c_j|,\quad D_{\max}=\max_{i=\overline{1,m}}|d_i|,\quad M\geq A_{\max}+C_{\max}+D_{\max}.$$

The last m columns of the restrictions form the unit matrix $I_{m\times m}$.

The classical linear program (6) is a minimum allowable primary program (PPA).

Step 2. Obtaining the initial Simplex table S_0

Now the constraint matrix contains n+k+m columns: $\tilde{\mathbf{A}}=(\tilde{a}_{ij})_{\substack{i=\overline{1,m}\\ j=\overline{1,n+k+m}}}$.

The program (PPA) in relations (6) contains a first ***allowable basic solution*** in which the first n+k variables are null (called ***additional variables***) and the last m variables (called ***basic variables***) are equal to the free terms:

$$\begin{cases} x_1=0=x_2=\ldots=x_{n+k} \\ x_{n+k+1}=d_1,\ x_{n+k+2}=d_2,\ \ldots,\ x_{n+k+m}=d_m \end{cases} \tag{7}$$

For the admissible solution (7) the objective function has the value:

$$z=M\cdot(d_1+d_2+\ldots+d_m)=M\cdot\sum_{i=1}^{m}d_i \tag{8}$$

The ***simplex table*** associated with the program (PPA) in the relations (6) is the matrix $\mathbf{S}=(s_{ij})_{\substack{i=\overline{1,m}\\ j=\overline{1,m+k+n+1}}}=(\overline{A}\,|\,D)$ composed of the extended matrix $\overline{A}$ (the initial matrix of the restrictions A to which the columns corresponding to the deviation variables and the penalization variables were added) and to which we have added to the right, as a last column, the vector of free terms $\mathbf{D^B}$.

For the easy algorithm calculations, the table is supplemented with another line containing the objective function coefficients.

The columns are noted (above) with the corresponding variables: $x_1, x_2, \ldots, x_{n+k+m}$, and the lines are named (left, in a new column **B**) with the corresponding basic variables:

Table 1

The initial simplex table S_0 associated with an allowable primary program (PPA)

B	$\mathbf{x_1}$	$\mathbf{x_2}$	**…**	$\mathbf{\downarrow x_q}$	**…**	$\mathbf{x_{n+k}}$	$\mathbf{x_{n+k+1}}$	$\mathbf{x_{n+k+2}}$	**…**	$\mathbf{x_{n+k+m-1}}$	$\mathbf{x_{n+k+m}}$	$\mathbf{D^B}$	$\mathbf{C^B}$
$\mathbf{c_j}$	c_1	c_2	…	c_q	…	c_n	M	M	M	M	M		
$\mathbf{x_{n+k+1}}$	s_{11}	s_{12}	…	s_{1q}	…	$s_{1,n+k}$	1	0	…	0	0	d_1	$c_1^B=M$
$\mathbf{x_{n+k+2}}$	s_{21}	s_{22}	…	s_{2q}	…	$s_{2,n+k}$	0	1	…	0	0	d_2	$c_2^B=M$
…	…	…	…	…	…	…	…	…	…	…	…	…	…
$\mathbf{\leftarrow x_p}$	s_{p1}	s_{p2}	…	$\boxed{s_{pq}}$	…	$s_{p,n+k}$	0	0	…	0	0	d_p	$c_p^B=M$

Table 1 (continued)

...	...	...	...	...	...	...	...	...	...	...	...	...	...
$\mathbf{x_{n+k+m-1}}$	s_{m-11}	s_{m-12}	...	s_{m-1q}	...	$s_{m-1,n+k}$	0	0	...	1	0	d_{m-1}	$c_{m-1}^{B}=M$
$\mathbf{x_{n+k+m}}$	s_{m1}	s_{m2}	...	s_{mq}	...	$s_{m,n+k}$	0	0	...	...	1	d_m	$c_m^{B}=M$
$\mathbf{z_j^B}$	z_1^B	z_2^B		z_q^B	...	...	...	...	...		...	...	
$\mathbf{\Delta_j = c_j - z_j^B}$	$c_1 - z_1^B$	$c_2 - z_2^B$		$c_q - z_q^B$	...	...	...	...	...		...	...	

The simplex table is also supplemented by a last column (on the right) containing the coefficients from the objective function corresponding to the basic variables ($\mathbf{C^B}$, has initially only M values).

Step 3. Obtaining the pivot position (p_r, q_r) in a ***Simplex table*** (initially r=0).

The column q_r and the line p_r correspond to the input variable and respectively the base output variable.

In order to determine them, the simplex table is completed with two more lines ($\mathbf{z_j^B}$ and Δ_j) calculated as follows:

$$z_j^B = \sum_{i=1}^{m} c_i^B \cdot s_{ij} \qquad (9)$$

$$\Delta_j = c_j - z_j^B \qquad , (\forall) j = \overline{1, n+k+m+1} \qquad (10)$$

The component $\mathbf{z_j^B}$ in column j is obtained by cumulating the products between the values in the last column ($\mathbf{C^B}$) and those in the respective column j (S_{ij}).

The differences in the last line (Δ_j) are obtained by subtracting from the first line (the objective function line) of the previous line ($\mathbf{z_j^B}$).

$\mathbf{x_q}$ and $\mathbf{x_p}$ are established by ***the input criteria*** and ***the output in / from the base*** respectively**.**

The non-basic variable $\mathbf{x_q}$ ***enters the base*** with the smallest non-positive difference:

$$\boxed{\Delta_q = \inf_{j \in S} \{\Delta_j \le 0\}} \qquad (11)$$

If ***all the differences are positive*** $(\Delta_j > 0 \; , \forall j \in S)$ then ***the*** basic ***program*** corresponding to the Simplex S_r ***is optimal*** and ***the optimal solution is unique.*** One can proceed to step 5.

If by entering the base of the variable $\mathbf{x_q}$ a previous base is repeated, then ***the program has multiple optimal solution***. One can proceed to step 5.

If the variable $\mathbf{x_q}$ enters the base, then the variable $\mathbf{x_p}$ exists with the lowest ratio:

$$\boxed{\frac{d_p}{s_{pq}} = \inf_{s_{iq} > 0} \left\{ \frac{d_i}{s_{iq}} \right\}, (\forall) i \in B} \qquad (12)$$

If $s_{iq} \le 0$,$(\forall) i \in B$ then ***the program has infinite optimal***. STOP.

Step 4. Switching from the Simplex table S_r ***to the next Simplex*** S_{r+1}: $S_r \to S_{r+1}$

The new simplex is obtained by ***the pivoting operation*** in the simplex $\mathbf{S^{(r)}} = (s_{ij}^{(r)})_{m\times(m+k+n+1)}$ with the pivot $s_{pq}^{(r)} \ne 0$ from the position (p_r, q_r), an operation that involves the following calculations:

– the pivot line is divided by the pivot (the pivot position is set to 1):

$$\boxed{s_{pj}^{(r+1)} = \frac{s_{pj}^{(r)}}{s_{pq}^{(r)}}} \quad ,\forall j \ne q \qquad s_{pq}^{(r+1)} = 1 \tag{13}$$

– the pivot column is filled in with 0: $s_{iq}^{(r+1)} = 0$ $\quad ,\forall i \ne p$

– any other element in the table is transformed according to ***the rule of the rectangle:***

$$\boxed{s_{ij}^{(r+1)} = \frac{s_{ij}^{(r)} \cdot s_{pq}^{(r)} - s_{iq}^{(r)} \cdot s_{pj}^{(r)}}{s_{pq}^{(r)}}} \quad ,i \ne p \;\; \text{or} \;\; j \ne q \tag{14}$$

After pivoting, in the new simplex $\mathbf{S^{(r+1)}}$ we change in column **B** the variable $\mathbf{x_p}$ with the variable $\mathbf{x_q}$ and in the last column $\mathbf{C^B}$ (on the line of $\mathbf{x_q}$) we change the coefficient with the corresponding one from the objective function (c_q).

r ($r+1 \to r$) is increased and ***one can return to step 3***.

Step 5. The reading of the optimal solution / solutions and the optimal value for the objective function, from the last simplex tables.

After having deleted from the last simplex table of all columns (keeping only two columns, **B** and $\mathbf{D^B}$) and after having deleted the first and last line (the objective function and difference line), a table of the following form is obtained:

B	$\mathbf{D^B}$
x_{1b}	d_{1u}
x_{2b}	d_{2u}
...	...
x_{mb}	d_{mu}
$\mathbf{z_j^B}$	z_u^B

The optimal solution and ***the optimal value*** are:

$$x_{1b}^* = d_{1u}, \quad x_{2b}^* = d_{2u}, \quad \ldots, \quad x_{mb}^* = d_{mu}, \qquad\qquad z^* = z_u^B$$

If at least one non-null non-penalty variable is entered in the last base, then ***the program has no permissible solutions*** (it was incorrectly stated).

we state that ***the algorithm cycles*** if ***the same base is obtained a second time***.

In this case, ***the program supports multiple optimal solutions.***

The last v simplex tables that have the same optimal value correspond to the v extreme optimal solutions:

$$X^{*(1)}=(x_1^{*(1)},x_2^{*(1)},...,x_n^{*(1)}),\ X^{*(2)}=(x_1^{*(2)},x_2^{*(2)},...,x_n^{*(2)}),\ ...,\ X^{*(v)}=(x_1^{*(v)},x_2^{*(v)},...,x_n^{*(v)})$$

The general optimal solution is the convex combination of these v extreme solutions:

$$X^*(\lambda_1,\lambda_2,...,\lambda_v)=\lambda_1\cdot X^{*(1)}+\lambda_2\cdot X^{*(2)}+...+\lambda_v\cdot X^{*(v)}$$
$$\forall\lambda_k\geq 0 \text{ cu } \lambda_1+\lambda_2+...+\lambda_v=1.$$

Finally, it should be remembered that applying ***the simplex algorithm with the penalty method*** can end with one of the following four possibilities:

a. ***the program has a unique optimal solution*** if all the differences in the columns of the non-basic variables are strictly positive ($\Delta_j>0\ ,\forall j\in S$).

b. ***the program has multiple optimal solutions*** if the same basis is obtained for the second time.

c. ***the program has infinite optimum*** if x_q enters the base and $s_{iq}\leq 0$, $,(\forall)i\in B$.

d. ***the program has no admissible solutions*** if in the last base there are also non-null penalty variables.

Solving fuzzy linear programs (PLF) with modified simplex algorithm

A fuzzy linear program (PLF) is a linear program in which the coefficients of the objective function, the technological matrix components and the free terms have uncertain mathematically modeled values with fuzzy numbers (only triangular fuzzy numbers will be used below):

$$(\text{PLF})\left|\begin{array}{l}[\max/\min]\ \tilde{z}(\tilde{C},\tilde{X})=\tilde{C}\cdot\tilde{X}=\sum_{j=1}^{n}\tilde{c}_j\tilde{x}_j\\ \hline \left[\begin{array}{l}\tilde{A}\cdot\tilde{X}\ \underset{\prec}{\overset{\succ}{=}}\ \tilde{D}\\ \tilde{x}_j\succeq\tilde{0}\qquad ,(\forall)j=\overline{1,n}\end{array}\right.\end{array}\right.\qquad(15)$$

The linear program (PLA) associated to a fuzzy linear program (PLF) is ***a classic linear program*** (in real numbers) obtained by replacing the triangular fuzzy numbers with their centers of gravity according to the corresponding relation (2).

$$\left.\begin{array}{l}\tilde{c}_j \longrightarrow <\tilde{c}_j> \overset{\text{not.}}{=} c_j\\ \tilde{a}_{ij} \longrightarrow <\tilde{a}_{ij}> \overset{\text{not.}}{=} a_{ij}\\ \tilde{d}_i \longrightarrow <\tilde{d}_i> \overset{\text{not.}}{=} d_i\\ \tilde{x}_j \longrightarrow <\tilde{x}_j> \overset{\text{not.}}{=} x_j\end{array}\right] \left| \begin{array}{l}, \forall i=\overline{1,m}\\ , \forall j=\overline{1,n}\end{array}\right. \Rightarrow$$

$$(\text{PLA}) \left| \begin{array}{l}[\max/\min]\ z(C,X) = C\cdot X = \sum_{j=1}^{n} c_j x_j\\ \hline \left[\begin{array}{l}A\cdot X \overset{\geq}{\underset{\leq}{=}} D\\ x_j \geq 0 \qquad ,(\forall) j=\overline{1,n}\end{array}\right.\end{array}\right. \tag{16}$$

Solving a fuzzy linear program (PLF) described by the relations (15) involves following the next algorithmic steps (***Modified Simplex Algorithm***):

PF1. ***All the centers of gravity*** of the input parameters ($\tilde{c}_j, \tilde{a}_{ij}, \tilde{d}_i$) are calculated, and ***the associated linear program (PLA)*** is obtained, which has only real number components.

PF2. The steps of ***the simplex algorithm with the penalty method*** are applied to the classical program (PLA):

– ***Step 1 and Step 2***. The permissible program (PPA) and the first simplex S_0 are established

– ***Step 3*** and ***Step 4*** are repeated k times.

The determining of the pivot positions [***(p_1,q_1), ... ,(p_k,q_k)***] involves calculating the differences Δ_j and reports $\frac{c_i^B}{s_{ij}}$.

By performing the k pivoting, the simplex tables are obtained: $S_0 \to S_1 \to ... \to S_k$.

PF3. One can return to the initial fuzzy program (PLF) to which a small Simplex table reduced $\tilde{S}_0$ is associated, where only the columns of variables that entered the base through the pivotals from step PF2 (columns ***q_1, ... , q_k***), column **B** (non-numerical).) and the $\mathbf{D^B}$ column are introduced.

The column $\mathbf{C^B}$, the first line and the last two (z_j^B and Δ_j) are no longer needed.

PF4. ***K pivoting movements are performed*** with the positions of the pivots ***(p_1,q_1), ... , (p_k,q_k)*** starting from ***the fuzzy simplex*** $\tilde{S}_0$ and fuzzy simplex tables are obtained: $\tilde{S}_0 \to \tilde{S}_1 \to ... \to \tilde{S}_k$.

At each pivoting the calculations (the divisions on the pivot line and the rectangle rule) are made with triangular fuzzy numbers as defined in the relations (3). In the new simplex fuzzy tables, the calculations in the columns corresponding to the pivots already used are given up.

PF5. The fuzzy components of ***the optimal solution / solutions*** from the last fuzzy Simplex tables ***are read***. In the case of the multiple solutions, the convex combination of the general optimal solution, dependent on v extreme solutions and the real subunit parameters $\lambda_v \in [0,1]$, is calculated.

The optimal value of the objective function (***fuzzy number***) and its center of gravity is calculated. STOP.

An example of solving a fuzzy linear program (PLF) with the modified simplex algorithm

A problem of production planning (hypothetical case)

In a trading company, for the production of the products P_1, P_2, and P_3, the raw materials M_1, M_2 and M_3 are used.

For the manufacture of product P_1 a_{11} units of the raw material M_1, a_{12} units of M_2, and a_{13} units of M_3 are used; for the manufacture of the product P_2 a_{21}, a_{22} and a_{23} units of raw materials are used and for the manufacture of the product P_3 a_{31} is used, a_{32} and a_{33} units of raw materials are used. In stock are d_1 units of raw material M_1, d_2 units of M_2 and d_3 units of M_3 are in stock.

The unit benefits are c_1 u.m. (monetary units) for P_1, c_2 u.m. for P_2 and c_3 u.m. for product P_3. What is the production plan (quantities x_1, x_2 and x_3 of products P_1, P_2 and P_3) in order to obtain the maximum profit?

Given that the general economic stability is very low, all the initial data defining the previous model represent profoundly uncertain information.

Thus, the unit benefit obtained for a P_1 product, which under conditions of total certainty would be 6 u.m, it is estimated by specialists to be between 4 and 10 u.m, as it is mathematically modeled with the triangular fuzzy number $\tilde{c}_1 = (4,5,10)_6$. Similarly, $\tilde{c}_2 = (6,7,8)_7$, $\tilde{a}_{11} = (4,5,6)_5$, etc.

The fuzzy linear program (PLF) corresponding to the presented economic problem has the following form:

$$
(\text{PLF})\quad \left[\begin{array}{l}
\underline{[\max]\ \tilde{z} = (4,5,10)_6 \cdot \tilde{x}_1 + (6,7,8)_7 \cdot \tilde{x}_2 + (5,10,11)_9 \cdot \tilde{x}_3} \\
\begin{cases}
(4,5,\ 6)_5 \cdot \tilde{x}_1 + (\ 3,\ 4,\ 5)_4 \cdot \tilde{x}_2 + (3,4,\ 9)_5 \cdot \tilde{x}_3 \le (3000,3100,3600)_{3200} \\
(4,5,10)_6 \cdot \tilde{x}_1 + (10,11,16)_{12} \cdot \tilde{x}_2 + (8,9,14)_{10} \cdot \tilde{x}_3 \le (5000,5100,5600)_{5200} \\
(3,4,\ 5)_4 \cdot \tilde{x}_1 + (\ 1,\ 2,\ 3)_2 \cdot \tilde{x}_2 + (2,7,\ 8)_6 \cdot \tilde{x}_3 \le (2200,2300,2800)_{2400}
\end{cases} \\
\qquad\qquad\qquad \tilde{x}_{1,2,3} \ge 0
\end{array}\right.
$$

The modified simplex algorithm is applied to solve the fuzzy linear program.

PF1. The first step involves calculating the gravity centers of the triangular fuzzy numbers:

$$<\tilde{c}_1> = <(4,5,10)> \overset{(2)}{=} \frac{4+2\cdot 5+10}{4} = 6, \ldots, \quad <\tilde{d}_3> = \frac{2200+2\cdot 2300+2800}{4} = 2400$$

In (PLF) the gravity centers were given as final indices.

After replacing the fuzzy numbers with their gravity centers, the associated classical program is obtained:

$$(\text{PLA}) \quad \left[\begin{array}{l} \underline{[\max]\ z = 6\cdot x_1 + 7\cdot x_2 + 9\cdot x_3} \\ \begin{cases} 5\cdot x_1 + \ \ 4\cdot x_2 + \ \ 5\cdot x_3 \le 3200 \\ 6\cdot x_1 + 12\cdot x_2 + 10\cdot x_3 \le 5200 \\ 4\cdot x_1 + \ \ 2\cdot x_2 + 6\cdot x_3 \le 2400 \end{cases} \\ \qquad\qquad x_{1,2,3} \ge 0 \end{array}\right.$$

PF2. ***The simplex algorithm with the penalty method*** *is applied to the classical program (PLA)*

Thus, after having passed the maximum to the minimum ($w = -z$) and adding the deviation unknowns (which are also penalty variables with M = 900 sufficiently large), we obtain the allowable primal program (PPA) and the corresponding Simplex S_0 table:

$$(\text{PPA}) \quad \left[\begin{array}{l} \underline{[\min]\ w = -6\cdot x_1 - 7\cdot x_2 - 9\cdot x_3 + 900\cdot(x_4 + x_5 + x_6)} \\ \begin{cases} 5\cdot x_1 + \ \ 4\cdot x_2 + \ \ 5\cdot x_3 + x_4 = 3200 \\ 6\cdot x_1 + 12\cdot x_2 + 10\cdot x_3 + x_5 = 5200 \\ 4\cdot x_1 + \ \ 2\cdot x_2 + 6\cdot x_3 + x_6 = 2400 \end{cases} \\ \qquad\qquad x_{1,2,\ldots,6} \ge 0 \end{array}\right.$$

Table 2

The Simplex algorithm for the program (PPA)

	B	x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{D^B}$	$\mathbf{C^B}$	
		–6	–7	–9	900	900	900			
	x_4	5	4	5	1	0	0	3200	900	3200/5=640
$\mathbf{S_0}$	x_5	6	12	10	0	1	0	5200	900	5200/10=520
	x_6	4	2	$\boxed{6}$	0	0	1	2400	900	2400/6=400
	z_j^B	13500	16200	18900	900	900	900	9720000	0	
	Δ_j	–13506	–16207	$\boxed{-18909}$	0	0	0			

Table 2 (continued)

		–6	–7	–9	900	900	900			
S_1	x_4	5/3	7/3	0	1	0	–5/6	1200	900	3600/7>500
	x_5	–2/3	$\boxed{26/3}$	0	0	1	–5/3	1200	900	3600/26<180
	x_3	2/3	1/3	1	0	0	1/6	400	–9	1200
	z_j^B	894	9897	–9	900	900	–4503/2	2156400	0	
	Δ_j	–900	$\boxed{-9904}$	0	0	0	6303/2			

		–6	–7	–9	900	900	900			
S_2	x_4	$\boxed{24/13}$	0	0	1	–7/26	–5/13	11400/13	900	11400/24<500
	x_2	–1/13	1	0	0	3/26	–5/26	1800/13	–7	
	x_3	9/13	0	1	0	–1/26	3/13	4600/13	–9	4600/9>500
	z_j^B	21526/13	–7	–9	900	-3156/13	–9019/26	10206000/13	0	
	Δ_j	– 21604/13	0	0	0	14856/13	32419/26			

		–6	–7	–9	900	900	900		
S_3	x_1	1	0	0	13/24	–7/48	–5/24	475	–6
	x_2	0	1	0	1/24	5/48	–5/24	175	–7
	x_3	0	0	1	–3/8	1/16	3/8	25	–9
	z_j^B	–6	–7	–9	–1/6	–5/12	–2/3	–4300	0
	Δ_j	0	0	0	5401/6	10805/12	2702/3		

Step 3 and Step 4 (from the classic Simplex algorithm) are repeated three times.

At the third iteration, the Simplex algorithm proceeds to step 5 because all the differences corresponding to the base variables are non-negative (in fact all three are null, 0).

Step 5. The program (PPA) has a unique solution:

$$\boxed{X^{*(1)} = (x_1^*, x_2^*, x_3^*) = (475, 175, 25)}$$

$$\boxed{w^* = -6 \cdot x_1^* - 7 \cdot x_2^* - 9 \cdot x_3^* + 900 \cdot (0+0+0) = -6 \cdot 475 - 7 \cdot 175 - 9 \cdot 25 = -4300}$$

$$\boxed{z^* = -w^* = -(-4300) = 4300}$$

The positions of the three pivots were: (3, 3), (2, 2) and (1, 1) respectively.

PF3. The reduced fuzzy Simplex program is $\widetilde{S}_0$ from table no. 3, a table containing only columns of variables $\mathbf{x_1}$, $\mathbf{x_2}$ and $\mathbf{x_3}$ and columns $\mathbf{B}$, $\mathbf{D^B}$. All columns and lines that do not participate in fuzzy pivoting operations have been removed:

Table 3

Fuzzy pivot operations in the modified simplex algorithm

$\tilde{S}_0$

B	**x_1**	**x_2**	**x_3**	**D^B**
x_4	4	3	3	3000
	5	4	4	3100
	6	5	9	3600
	5	4	5	3200
x_5	4	10	8	5000
	5	11	9	5100
	10	16	14	5600
	6	12	10	5200
x_6	3	1	2	2200
	4	2	7	2300
	5	3	8	2800
	4	2	6	2400

$\tilde{S}_1$

B	**x_1**	**x_2**	**x_3**	**D^B**
x_4	–0.847	0.097		–266.667
	2.181	2.778		1529.167
	3.153	3.681		2008.333
	1.667	2.333		1200
x_5	–3.361	2.528		–683.333
	–0.556	9.722		1633.333
	1.806	12.694		2216.667
	–0.667	8.667		1200
x_3	0.361	0.139		250
	0.722	0.361		425
	0.861	0.472		500
	0.667	0.333		400

$\tilde{S}_2$

B	**x_1**	**x_2**	**x_3**	**D^B**
x_4	0.059			–127.885
	2.139			1004.604
	3.048			1626.368
	1.846			876.923
x_2	–0.25			–19.231
	–0.075			171.893
	0.093			229.29
	–0.077			138.462
x_3	0.225			105.609
	0.768			388.696
	1.008			532.384
	0.692			353.846

All numeric cells contain 4 components: the first three are the components of the triangular fuzzy number and the fourth (below the dotted line) is its center of gravity.

$\widetilde{S}_3$

B	x_1	x_2	x_3	D^B
x_1				–27.008 547.237 832.534
				475
x_2				–13.256 203.845 305.566
				175
x_3				–242.506 28.747 285.012
				25

PF4. The calculations of the three pivoting movements containing operations with triangular fuzzy numbers are as follows:

The first pivot movement that goes from the table (simplex fuzzy) $\widetilde{S}_0$ to the table $\widetilde{S}_1$ has the fuzzy number $\widetilde{a}_{33}^{(0)} = (2,7,8)_6$ as pivot. The column of the pivot is no longer calculated.

The elements on the pivot line (line 3) are divided by the pivot:

$$\widetilde{s}_{31}^{(1)} = \frac{\widetilde{s}_{31}^{(0)}}{\widetilde{s}_{33}^{(0)}} = \frac{(3,4,5)_4}{(2,7,8)_6} = \frac{6\cdot(3,4,5)+4\cdot(2,7,8)}{2\cdot 6^2} = \frac{(18+8,24+28,30+32)}{72} = \left(\frac{13}{36},\frac{26}{36},\frac{31}{36}\right)_{\frac{2}{3}}$$

$$\widetilde{s}_{31}^{(1)} \approx (0.361, 0.722, 0.861)_{0.667}$$

The center of gravity, given as an index, is:

$$<\widetilde{s}_{31}^{(1)}> = \frac{13+2\cdot 26+31}{4\cdot 36} = \frac{96}{4\cdot 36} = \frac{2}{3} \approx 0.667\cdot$$

The other two fuzzy numbers on the pivot line are similarly calculated:

$$\widetilde{s}_{32}^{(1)} = \frac{\widetilde{s}_{32}^{(0)}}{\widetilde{s}_{33}^{(0)}} = \frac{(1,2,3)_2}{(2,7,8)_6} = \ldots \quad \text{respectiv} \quad \widetilde{d}_3^{(1)} = \frac{\widetilde{d}_3^{(0)}}{\widetilde{s}_{33}^{(0)}} = \frac{(2200,2300,2800)_{2400}}{(2,7,8)_6} = \ldots$$

The other six fuzzy numbers that make up the simplex $\widetilde{S}_1$ are calculated according to the rectangle rule:

$$\widetilde{s}_{11}^{(1)} = \frac{\widetilde{s}_{11}^{(0)}\cdot\widetilde{s}_{33}^{(0)} - \widetilde{s}_{31}^{(0)}\cdot\widetilde{s}_{13}^{(0)}}{\widetilde{s}_{33}^{(0)}} = \frac{(4,5,6)_5\cdot(2,7,8)_6 - (3,4,5)_4\cdot(3,4,9)_5}{(2,7,8)_6}$$

$$(4,5,6)_5\cdot(2,7,8)_6 = \frac{6\cdot(4,5,6)+5\cdot(2,7,8)}{2} = \frac{(24+10,30+35,36+40)}{2} = \frac{(34,65,76)_{60}}{2}$$

$$(3,4,5)_4 \cdot (3,4,9)_5 = \frac{5\cdot(3,4,5)+4\cdot(3,4,9)}{2} = \frac{(27,36,61)_{40}}{2}$$

$$\frac{(34,65,76)_{60}}{2} - \frac{(27,36,61)_{40}}{2} = \frac{(34-61,65-36,76-27)}{2} = \frac{(-27,29,49)_{20}}{2}$$

$$\widetilde{s}_{11}^{(1)} = \frac{\frac{(-27,29,49)_{20}}{2}}{(2,7,8)_6} = \frac{6\cdot(-27,29,49)+20\cdot(2,7,8)}{2\cdot 2\cdot 6^2} = \frac{(-162+40,174+140,294+160)}{4\cdot 36}$$

$$\widetilde{s}_{11}^{(1)} = \frac{(-122,314,454)}{144} = \frac{(-61,157,227)_{120}}{72} \approx (-0.847,2.181,3.153)_{1.667}$$

The other five fuzzy numbers in the simplex are calculated similarly (according to the rectangle rule):

$$\widetilde{s}_{12}^{(1)} = \frac{\widetilde{s}_{12}^{(0)}\cdot\widetilde{s}_{33}^{(0)} - \widetilde{s}_{32}^{(0)}\cdot\widetilde{s}_{13}^{(0)}}{\widetilde{s}_{33}^{(0)}} = \ldots,\quad \widetilde{s}_{21}^{(1)} = \frac{\widetilde{s}_{21}^{(0)}\cdot\widetilde{s}_{33}^{(0)} - \widetilde{s}_{31}^{(0)}\cdot\widetilde{s}_{23}^{(0)}}{\widetilde{s}_{33}^{(0)}} = \ldots,$$

$$\widetilde{s}_{22}^{(1)} = \frac{\widetilde{s}_{22}^{(0)}\cdot\widetilde{s}_{33}^{(0)} - \widetilde{s}_{32}^{(0)}\cdot\widetilde{s}_{23}^{(0)}}{\widetilde{s}_{33}^{(0)}} = \ldots$$

$$\widetilde{d}_1^{(1)} = \frac{\widetilde{d}_1^{(0)}\cdot\widetilde{s}_{33}^{(0)} - \widetilde{s}_{13}^{(0)}\cdot\widetilde{d}_3^{(0)}}{\widetilde{s}_{33}^{(0)}} = \ldots,\qquad \widetilde{d}_2^{(1)} = \frac{\widetilde{d}_2^{(0)}\cdot\widetilde{s}_{33}^{(0)} - \widetilde{s}_{23}^{(0)}\cdot\widetilde{d}_3^{(0)}}{\widetilde{s}_{33}^{(0)}} = \ldots$$

The first iteration of the Fuzzy simplex algorithm is fully achieved: $\widetilde{S}_0 \longrightarrow \widetilde{S}_1$.

The calculations for the other two iterations are similar $\widetilde{S}_1 \longrightarrow \widetilde{S}_2 \longrightarrow \widetilde{S}_3$.

PF5. The last Fuzzy simplex table $\widetilde{S}_3$ reads the following solutions:

$$\boxed{\widetilde{x}_1^* = (-27.008, 547.237, 832.534)_{475}} \qquad \boxed{\widetilde{x}_2^* = (-13.256, 203.845, 305.566)_{175}}$$

$$\boxed{\widetilde{x}_3^* = (-242.507, 28.747, 285.012)_{25}}$$

The optimal value and its center of gravity are:

$$<\widetilde{w}^*> = <-(4,5,10)_6\cdot\widetilde{x}_1^* - (6,7,8)_7\cdot\widetilde{x}_2^* - (5,10,11)_9\cdot\widetilde{x}_3^*>$$

$$<\widetilde{w}^*> = -6\cdot 475 - 7\cdot 175 - 9\cdot 25 = -4300,\ \boxed{<\widetilde{z}^*> = -<\widetilde{w}^*> = -(-4300) = 4300}.$$

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